Endogenous business cycle dynamics within the inventory model of Metzler: Adding an inventory floor

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Abstract

The inventory model of Metzler may produce dampened fluctuations in economic activity and thus contributes to our understanding of business cycle dynamics. For some parameter combinations, however, the model generates oscillations with increasing amplitude, implying that the inventory stock of the firms eventually turns negative. Taking this observation into account, we reformulate Metzler's model by simply putting a floor to the inventory level. Within the new piecewise-linear model, endogenous business cycle dynamics may now be triggered via a center bifurcation, i.e. for certain parameter combinations production changes are (quasi-)periodic.

Keywords

Inventory adjustments; business cycles; piecewise-linear model; center bifurcation.

JEL classification

E12; E32.

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1 Introduction

Linear difference equation models of the business cycle may produce dampened oscillations. Presumably the most prominent contribution in this field is the multiplier-accelerator model of Samuelson (1939). However, these difference equation approaches may also yield cycles with increasing amplitude, implying that national income and related economic variables are driven away from their equilibrium values and eventually become negative. Hicks (1950) thus suggested adding boundaries for some economic variables to such models. In particular, his focus was on the investment part of the multiplier-accelerator model for which he introduced a floor and a ceiling. As a result, his model has the potential to generate bounded dynamics. From a mathematical point of view, the Hicks model corresponds to a piecewise-linear model. Recently, such models have regained some interest (e.g. Gallegatti et al. 2003, Puu et al. 2005, Gardini et al. 2006, Puu 2006) since new mathematical tools, as presented in Sushko and Gardini (2006), have been developed to analyze them.

The goal of our paper is to provide a quite natural reformulation of Metzler's (1941) model.[1] A key building block of this renowned model is the inventory stock of the firms, a quantity which obviously cannot be negative. By adding a floor to the inventory stock, we are able to rewrite Metzler's model as a piecewise-linear difference equation model. A key result of our paper is that the new model has only one economically meaningful fixed point which may loose its local stability only via a so-called center bifurcation, after which distinct (quasi-)periodic business cycles set in.

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[1] Although Metzler formulated his model already in 1941, it still attracts a fair share of attention. For instance, Eckalbar (1985), Zhang (1989), Franke and Lux (1993), Matsumoto (1998) and Chiarella et al. (2005) discuss interesting nonlinear extensions of Metzlers approach. General surveys of nonlinear dynamic models in economics are provided by Day (1999) and Rosser (2000).

The paper is organized as follows. In section 2, we recall the inventory model of Metzler. In section 3, we modify the model and present our results. The last section provides some conclusions.

2 The inventory model of Metzler with naive expectations

In this section, we briefly summarize the model of Metzler (1941). A comprehensive analytical treatment may be found in Gandolfo (2005).[2] Metzler assumes that the producers desire to keep inventory proportional to expected sales of consumption goods. This may have some important consequences. Suppose, for instance, that the economy enters a recession so that consumer demand shrinks. Besides reducing production of consumption goods, the firms may further decide to cut their inventory stock, i.e. they produce even less than the consumers demand. Obviously, this may deepen the recession. The opposite occurs in an upswing where the firms produce more than the consumers demand. Adjustment of the inventory stock may thus amplify business cycles.

Let us now turn to the details of the model. The firms' total production level Y in period t is the sum of the three components

$$Y_t = I_t + U_t + S_t , \tag{1}$$

where I stands for current investment goods, U for current expected consumer demand and S for current inventory adjustments, respectively. Note that S may be positive (accumulation of inventory) or negative (depletion of inventory).

[2] Although the inventory of firms is a relatively small component of national income, the importance of changes in the inventory level has long been recognized (see, e.g. Binder and Maccini 1991 and Ramey and West 1999).

The production of investment goods is simply determined by

$$I_t = \bar{I}\ , \tag{2}$$

i.e. the production of investment goods is constant.

Expected sales in period t are given by last period's sales

$$U_t = C_{t-1}\,, \tag{3}$$

implying that the firms form naïve expectations. Metzler (1941) and Gandolfo (2005) also consider other behavioral expectation formation rules. However, for our purpose this is not necessary and therefore we confine ourselves to the simplest framework.

The consumption function is written as

$$C_{t-1} = bY_{t-1}\,. \tag{4}$$

Consumption in period $t-1$ is proportional to the consumers' income in that period. The marginal propensity to consume is restricted to $0 < b < 1$.

The adjustment of inventory stock is defined as

$$S_t = \hat{Q}_t - Q_{t-1}\,, \tag{5}$$

where $\hat{Q}$ stands for the desired amount of inventory and Q for the inventory level. For instance, if the inventory level is below its desired level, firms increase their production.

The desired inventory level of firms is proportional to expected sales

$$\hat{Q}_t = kU_t\,. \tag{6}$$

The parameter k, which may be regarded as an inventory accelerator, is positive.

Note that firms may not correctly predict consumer demand and thus may not realize their desired inventory stock. The inventory stock in period $t-1$ is given as

$$Q_{t-1} = \hat{Q}_{t-1} - (C_{t-1} - U_{t-1})\,. \tag{7}$$

If the firms have been too optimistic (pessimistic) with respect to consumer demand, the inventory stock is higher (lower) than the desired inventory stock.

Combining (1) to (7) delivers

$$Y_t - b(2+k)Y_{t-1} + b(1+k)Y_{t-2} = \bar{I} , \quad (8)$$

i.e. the development of production is due to a second-order linear difference equation.

Setting $\bar{Y} = Y_t = Y_{t-1} = Y_{t-2}$, we find the fixed point of (8) as

$$\bar{Y} = \frac{1}{1-b}\bar{I} , \quad (9)$$

which is identical to the traditional Keynesian multiplier solution.

Recall that the fixed point of a second-order linear difference equation $Y_t + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} = \beta$ is stable if the inequalities $1+\alpha_1+\alpha_2 > 0$, $1-\alpha_1+\alpha_2 > 0$ and $1-\alpha_2 > 0$ jointly hold. Straightforward calculations reveal that only the third inequality may be violated. The critical condition is

$$b < \frac{1}{1+k} \ . \quad (10)$$

Suppose, for instance that $b = 0.9$. Then the fixed point (9) looses its stability when k exceeds $k \approx 0.111$.

Moreover, a second-order linear difference equation generates oscillations if $\alpha_1^2 < 4\alpha_2$. For our model, this yields

$$b < \frac{4(1+k)}{(2+k)^2} . \quad (11)$$

Hence, if (10) and (11) hold, the model of Metzler generates business cycles with decreasing amplitude.

This case is illustrated in the top line of the panels of figure 1 where we assume that $b = 0.9$, $k = 0.1$ and $\bar{I} = 1$. The dynamics is plotted for 150 time steps. The left-hand panel depicts the evolution of national income and the right-hand panel presents the development of the firms' inventory. As can be seen, both quantities fluctuate around their equilibrium values and eventually approach them as time proceeds. However, if only (11) holds, the model generates oscillations with increasing amplitude, as revealed in the central line of panels where k is now 0.15 (the other parameters have not been changed). The right-hand panel shows that the inventory stock quickly takes on negative values, which, in reality, is not possible.

For later analysis it is important to note that in (b,k) -parameter space, condition (10) is always located below condition (11). For low values of b and k, the model thus always generates cycles with decreasing amplitude. If one or both parameters increase sufficiently enough, (10) gets violated and we observe cycles with increasing amplitude. If one or both parameters increase even further, (11) is also eventually violated and production monotonically explodes. Ignoring parameter combinations which lie exactly on these boundaries, the model thus yields a fixed point which is a stable focus, an unstable focus or an unstable node, respectively.

3 A reformulation of Metzler's inventory model

Since the inventory of the firms cannot become negative, we rewrite (7) as

$$Q_{t-1} = \begin{cases} 0 & \text{for} \quad \hat{Q}_{t-1} - (C_{t-1} - U_{t-1}) < 0 \\ \hat{Q}_{t-1} - (C_{t-1} - U_{t-1}) & \text{otherwise} \end{cases}, \tag{12}$$

i.e. the inventory now contains a floor.

As a result, the recurrence relation that determines production turns into

$$Y_t = \begin{cases} b(1+k)Y_{t-1} + \bar{I} & \text{for} \quad Y_{t-1} > (1+k)Y_{t-2} \\ b(2+k)Y_{t-1} - b(1+k)Y_{t-2} + \bar{I} & \text{otherwise} \end{cases}, \tag{13}$$

which is a second-order piecewise-linear difference equation.

Note that (13) has two fixed points of which only one is economically meaningful. To see this, consider first the upper branch of (13). Setting $\bar{Y} = Y_t = Y_{t-1}$ delivers $\bar{Y} = \bar{I}/(1-b(1+k))$. However, taking into account the condition for this branch we immediately see that this (saddle) fixed point only exists if it is negative (any positive fixed point is inconsistent with this condition). The lower branch of (13) is just the Metzler model and the fixed point (9) does not violate the condition for this branch. So, also the new model has only one economically meaningful fixed point.

Furthermore, this fixed point is locally stable as long as (10) holds and minor perturbations of the fixed point always trigger dampened cycles. If (10) is crossed, however, a pair of complex conjugate eigenvalues exits the unit circle. At this point, a so-called center bifurcation emerges after which (quasi-)periodic motion sets in (see Sushko and Gardini 2006 for more details).

This interesting phenomenon is illustrated in the bottom line of the panels of figure 1 in which we assume the same parameter setting as in the central line of the panels but now the model includes an inventory floor. As we can see, after an initial shock the dynamics is characterized by cycles with increasing amplitude, yet when the floor is hit, the amplitude starts to remain relatively constant and output does not become negative. A key result of our paper thus is that the trivial model of Metzler, simply buffeted with an inventory floor, may generate endogenous business cycles via a center bifurcation.

To explore this phenomenon in more detail, figure 2 presents three bifurcation diagrams in which we increase the parameter k from 0 to 0.2 in 200 discrete steps. We assume that $\bar{I} = 1$ but vary b from 0.95 (top), to 0.9 (central) and to 0.85 (bottom). The dynamics is plotted after a quite large transient of 15000 observations. Such bifurcation diagrams give a first helpful numerical impression of the behavior a dynamic model may produce as one of its parameters is changed.

What are the results of this exercise? First, as predicted by (10), the fixed point of the model looses its local stability when k exceeds a certain threshold. Using (10), we find that these critical values are 0.0526, 0.1111 and 0.1764, respectively, which is nicely conformed by figure 2. Hence, the larger the marginal propensity to consume, the lower the critical value of k which ensures local stability of the fixed point.

Second, all three panels reveal that the model may generate endogenous dynamics over an extended range of k. For $b = 0.95$, for instance, we observe quasi-periodic motion when k is located in the interval $0.052 < k < 0.135$. For some values of k, however, bounded oscillations appear where production is negative. If k exceeds 0.135, the system even explodes.

Third, when the center bifurcation occurs, the dynamics of the model changes fundamentally. Immediately after the fixed point has lost its local stability, we see cycles with substantial amplitudes. The transition from fixed point to (quasi-)periodic dynamics is clearly not smooth (as opposed to the related Neimark-Sacker bifurcation scenario). This may be of some policy importance: Even a modest change of one of the behavioral parameters may have dramatic consequences for the economy. Production becomes unstable and instead we observe pronounced business cycles.

Finally, figure 3 presents a double bifurcation diagram in which we vary parameter b between 0.8 and 1 and parameter k between 0 and 0.4. The parameter space which yields a stable fixed point is indicated by red. Similarly, the light gray area comprises parameter combinations which lead to an explosion. All other parameter combinations produce endogenous dynamics. However, the dark gray area is associated with dynamics where production is also negative. In this sense, the economically interesting part is given with the parameter space between the red and the dark gray area. The white area stands for cycles with a period length larger than 42 or quasi-periodic motion. The other colors indicate oscillations with lower cycle lengths. Exactly in this parameter space, the simple model of Metzler, buffeted with an inventory floor, yields interesting endogenous business cycle dynamics. This may in particular be relevant when the marginal propensity to consume is relatively high.

4 Conclusions

The inventory model of Metzler may generate dampened business cycle dynamics. In this paper, we reformulate this model by adding a floor to the inventory stock. This modification appears to be quite natural and leads to the following results. The new model has one interesting fixed point which is identical to the original model. This fixed point may loose its local stability only via a center bifurcation after which endogenous dynamics in the form of (quasi-)periodic motion sets in. The transition from fixed point dynamics to fluctuations with quite large amplitudes occurs abruptly. Since endogenous dynamics may be observed for a broader range of parameters, the modified Metzler model illustrates once again the importance of inventory adjustments for the emergence of endogenous business cycles. In addition, the model is another example of a piecewise-linear map which may lead to interesting bifurcation phenomena. One may extend our simple model by considering that also production may not become negative and by taking an additional capacity constraint into account. In order to make our point clear, however, we leave such extensions for future work.

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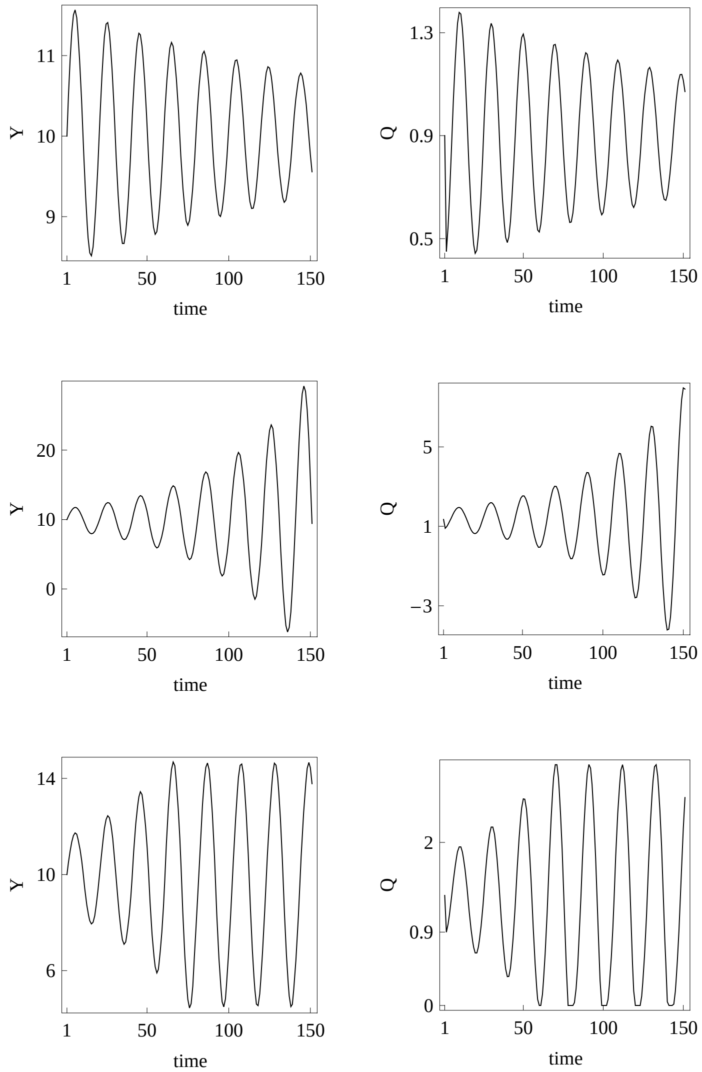

Figure 1: The evolution of production (left-hand panels) and inventory (right-hand panels) for different parameter combinations in the time domain. Top panel: $b = 0.9$, $k = 0.1$ and $\bar{I} = 1$. Central panel: $b = 0.9$, $k = 0.15$ and $\bar{I} = 1$. Bottom panel: $b = 0.9$, $k = 0.15$, $\bar{I} = 1$ and an inventory floor of 0.

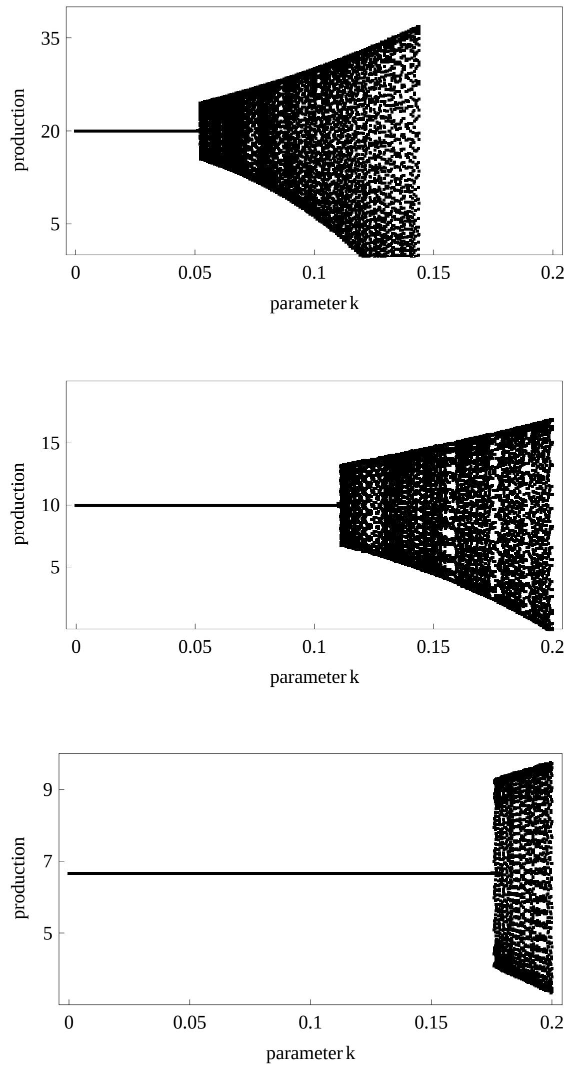

Figure 2: Bifurcation diagrams for the parameter k which is increased from 0 to 0.2 in 200 discrete steps. The dynamics is plotted after a transient of 15000 observations. Top: $b = 0.95$ and $\bar{I} = 1$. Central: $b = 0.9$ $\bar{I} = 1$. Bottom: $b = 0.85$ and $\bar{I} = 1$.

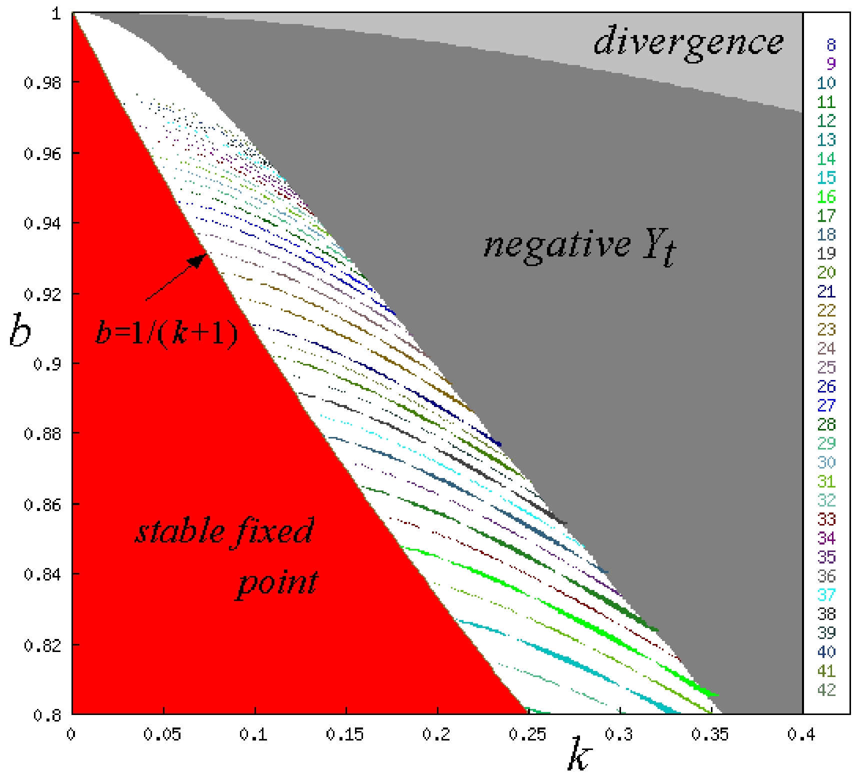

Figure 3: A double bifurcation diagram in which parameter b is varied between 0.8 and 1 and parameter k between 0 and 0.4. The parameter space which yields a stable fixed point (explosion) is indicated by red (light gray). The dark gray area is associated with dynamics where production is positive and negative. The parameter space between the red and the dark gray area generates dynamics where production is always positive. The white area stands for cycles with a period length larger than 42 or quasi-periodic motion. The other colors indicate lower cycle lengths.