# Online Appendix: Arbitrage-Free Smile Construction on FX Option Markets Using Garman-Kohlhagen Deltas and Implied Volatilities

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# A No-Arbitrage Condition for Premium Adjusted Deltas

The premium adjusted spot and forward delta $\Delta_{S,p.a}$ and $\Delta_{F,p.a}$ are given by

$$\begin{aligned}\Delta_{S,p.a.} &= cB_f(t,T)\frac{X}{F(t,T)}\Phi(cd_2) \\ \Delta_{F,p.a.} &= c\frac{X}{F(t,T)}\Phi(cd_2) = \frac{\Delta_{S,p.a.}}{B_f(t,T)}.\end{aligned} \tag{1}$$

Intuitively, the premium adjusted spot delta defines the amount of foreign currency that must be purchased to replicate the option in excess of the option premium. The premium adjusted forward delta assumes hedging with forward contracts. Premium adjusted deltas are used when option prices are paid in foreign currency. A prominent example is the USDJPY exchange rate: Option premiums are expressed in USD though the USD is the foreign currency.

Define an alternative moneyness variable $\tilde{d}$, which is related to premium adjusted put deltas $\Delta^P_{S,p.a.}$ per Equation (1) as

$$\tilde{d} = \Phi^{-1}\left(-\frac{\Delta^P_{S,p.a.}}{B_f(t,T)}\frac{F_t}{X^P}\right) = \Phi^{-1}\left(-\Delta^P_{F,p.a.}\frac{F_t}{X^P}\right). \tag{2}$$

Let $X_C$ and $X_P$ be the strikes of the call and put option with premium adjusted deltas $\Delta^C_{S,F,p.a.}(X_C) = \bar{\Delta}$ and $\Delta^P_{S,F}(X_P) = -\bar{\Delta}$ of a $\bar{\Delta}$-risk reversal/butterfly. The delta of the put option with strike $X_P$ is given by

$$\Delta^P_F(X_P) = \frac{\Delta^P_S(X_P)}{B_f(t,T)} = -\Phi\left(\tilde{d}\right)\frac{X_P}{F_t}, \tag{3}$$

where $\tilde{d} = -d_2$ follows from Equation (3) in the paper. Then the delta of the put option with strike $X_C$ is calculated as

$$\Delta^P_{F,p.a.}(X_C) = \frac{\Delta^P_{S,p.a.}(X_C)}{B_f(t,T)} = \frac{X_P\Phi\left(\tilde{d}\right) - X_C}{F_t}. \tag{4}$$

Considering the no-arbitrage condition in Equation (9) in the paper by Carr and Wu (2016), the Proportional Volatility Dynamics in Equation (10) in the paper, and the change of variables

$$\ln\left(\frac{X}{F}\right) = \sigma\sqrt{\tau}d - \frac{1}{2}\sigma^2\tau. \tag{5}$$

Then the implied volatilities are give by the roots of the cubic polynomial

$$\begin{aligned} g_{\tilde{d}}(\sigma) &= \alpha\sigma_t^3 + \beta\sigma_t^2 + \gamma\sigma_t + \delta = 0 \\ \alpha &= \tilde{d}\tau^{3/2}\xi_t^2 \\ \beta &= 1 - 2\theta_t\tau - \tilde{d}^2\tau\xi_t^2 \\ \gamma &= -2\tilde{d}\sqrt{v_t}\rho_t\sqrt{\tau}\xi_t \\ \delta &= -v_t. \end{aligned} \tag{6}$$

This cubic polynomial also has the symmetry property

$$g_{\tilde{d}}(\sigma_t) = g_{-\tilde{d}}(-\sigma_t). \tag{7}$$

The variable $\tilde{d}$ characterizes the moneyness of a put option underlying a $\bar{\Delta}$-risk reversal/butterfly. However and per Equation (4), the implied volatility of a call option underlying the same contract is not identified by $-\tilde{d}$ when deltas are premium adjusted. Hence, the roots of two cubic polynomials have to be considered to price the benchmark contracts risk reversal/butterfly.

According to the DNS-convention for ATM-volatilities, the moneyness variable $\tilde{d} = 0$ for

ATM-options. In this case, the cubic polynomial becomes

$$g_{\tilde{d}}(\sigma_t) = (1 - 2\theta_t \tau)\,\sigma_t^2 - v_t = 0 \tag{8}$$

Setting $v_t = \sigma_{ATM}^2$ imposes the constraint $\theta_t = 0$, which insures that ATM-volatilities are matched.

# References

Carr, P., & Wu, L. (2016). Analyzing volatility risk and risk premium in option contracts: A new theory. *Journal of Financial Economics*, *120*(1), 1–20. doi: https://doi.org/10.1016/j.jfineco.2016.01.004